Spatial Search*

Xiaoming Cai† Pieter Gautier‡ Ronald Wolthoff§

January 29, 2025

Abstract

This paper considers a random search model where some locations provide sellers with better chances of meeting many buyers than other locations (for example popular shopping streets or the first page of a search engine). When sellers are heterogeneous in terms of the quality of their product and/or the probability that a given buyer likes their product, it is desirable that sellers of high-quality niche products sort into the best locations (positive assortative matching, PAM). We show that this does not always happen in a decentralized market. Finally, we endogenize the location distribution and show that PAM between sellers and locations always arises in equilibrium. However, the equilibrium distribution of locations is too favorable for the sellers of high-quality, niche products.

JEL classification: C78, D44, D83.
Keywords: search frictions, spatial equilibrium, sorting

*We thank Sephorah Mangin, Guido Menzio and participants at the NYU 2023 Search and Matching conference for insightful comments. Cai thanks the Guangdong Natural Science Foundation for its support (Grant No. 2022A1515010982). Wolthoff gratefully acknowledges financial support from the Social Sciences and Humanities Research Council.

†Peking University HSBC Business School. E-mail: xmingcai@gmail.com.
‡VU University Amsterdam and Tinbergen Institute. E-mail: p.a.gautier@vu.nl.
§University of Toronto. E-mail: ronald.p.wolthoff@gmail.com.

1 Introduction

Good locations for sellers are locations where it is easy to meet buyers. They are often located in densely populated areas (Fifth Avenue in New York, Bond Street London) or they are easily reachable (good parking facilities or near a metro station). For online products, we can think of good locations as being on a top position of a search engine. In this paper, we consider a random search model that takes into account that some locations are better than others. The model offers a framework to study what type of sellers benefit from good locations and how this shapes spatial sorting, how the price of locations is determined and from an urban planning perspective, when is it desirable to make locations similar (in terms of meetings) and when it is better to have heterogeneity in location quality.

In our model, sellers are characterized by the quality of their product, denoted by z, and the probability that a buyer likes their product, which is assumed to be weakly decreasing in z and is denoted by $x(z)$.[1] Furthermore, we assume for simplicity that sellers have one good for sale, but what matters is only that there is some capacity constraint.[2] Locations differ in how many buyers there are per seller, taking into account a buyer-resource constraint. This implies that if we improve the meeting rate for sellers in one location (for example by adding a railway station), the sellers in other locations will meet fewer buyers. In each location, we have a constant-returns-to-scale Poisson meeting technology, but the queue length varies across locations. Formally, we model seller locations as points on a unit circle where uniformly distributed buyers move clockwise to the nearest location. A good location for sellers is then one which is far away from their nearest competitor in a counterclockwise direction.

We start with a simple environment where the distribution of locations is exogenous, and let sellers sort into their optimal location, given a competitive rental market. We show that whenever the expected consumer value $zx(z)$ is increasing in z, the planner would always prefer to match good locations with high-quality sellers, i.e., positive assortative matching (PAM) between sellers and locations is desirable. Next, we consider the decentralized market equilibrium, and assume that sellers use auctions as the selling mechanism. We show that in the decentralized market, the requirement for PAM is more stringent and a *sufficient* condition is that $zx^2(z)$ is increasing in z, since sellers enjoy a high payoff when there are two or more buyers who want to buy their product. Good locations are particularly valuable for high-quality goods that few buyers like but those who like it, like it a lot. This is consistent with the designer shops that are located on Fifth Avenue and the fact that most

[1] As in Kiyotaki and Wright (1993), more specialized goods are liked by fewer buyers but those who like it, like it a lot. They show that more specialized goods are offered when search frictions get smaller, which necessitates the use of money (instead of barter trade).

[2] A second-hand car seller with five cars of a particular type would like to meet at least six buyers to be able to be on the short-side of the market. Here a seller with one unit would like to meet at least two buyers and for both cases this is more likely to happen in a good location than in a bad location.

of the Michelin three-star restaurants are located in or near big cities.[3] Auctions serve as the selling mechanism in the benchmark model for the following reasons. First, this is the most profitable mechanism for sellers and we know that it generates efficiency when there is just one location (so any inefficiencies we find come from our new environment). Second, the intuition behind our results is more clear and simple with auctions than with other mechanisms. At the same time, in real world markets, we often observe price posting, so we will also analyze that case.

We find that although the total number of meetings is maximized when sellers are located at equidistance, this is not always the welfare-maximizing topography. For example, if buyers are located randomly and uniformly on the circle and a small fraction of them has a strong desire for top quality food, then there is no area where it is profitable for top restaurants to enter and even if we would force them to enter, they would create little surplus because they meet relatively few buyers. In contrast, when there is heterogeneity in locations, sellers of high-quality and/or niche products are able to create a lot of value by locating in good spots. So heterogeneous locations can generate more welfare than equidistant locations by allowing for sorting of seller types. This results in fewer trades overall but more high-quality trades.

Interestingly, even when search is random, heterogeneity in locations can still create heterogeneity in expected numbers of buyers per seller as in directed search models. Random search is relevant for settings where full ex-ante commitment is not possible. For example, a seller may announce a positive reserve price ex ante but ex post after one or more buyers visit, the seller has no incentive to reject values below the reserve price but above the seller's valuation. When search is directed, a seller can increase the expected number of buyers by offering a good deal to the buyers whereas here they can select a good location. An important difference is that here, the price for a good location does not go to the buyers, which can lead to inefficiencies in spatial sorting that are absent in a standard directed search model. We show that this does not only hold for the auction mechanism but also for price posting.

Next, we take an urban planning or regional policy perspective and consider the optimal distribution of locations given the distribution of seller types. We can think of regional policy in our framework as choosing the optimal topography which must strike a balance between maximizing trade and allowing high-quality sellers to sort into good locations. We show that the optimal topography mimics the directed search equilibrium, which equalizes buyers' marginal contribution to surplus in different submarkets. In contrast, in the random search equilibrium with an endogenous location distribution, high-quality sellers typically overinvest in good locations for rent seeking reasons (they are willing to pay higher rents to receive two

[3] Another example comes from the following Financial Times article that points out that most manufacturing has moved out of cities, except high-quality niche products that are produced in small quantities like micro-breweries, furniture makers, roboticists and 3D-printing specialists.

or more buyers rather than one). This is socially wasteful because a seller who meets two effective buyers creates the same surplus as a seller who meets one effective buyer. Moreover, a seller who invests in a good location and meets many buyers does not internalize that in other areas more sellers will meet no buyers at all.

Our model is consistent with the finding in Neiman and Vavra (2023) that niche consumption is largest in areas with many buyers per seller like Chicago, Washington DC, Tampa, Los Angeles and Boston and lowest in non-dense, isolated places like Des Moines, Little Rock, Las Vegas, and "West Texas". In the context of the labor market, Gautier and Teulings (2003) create an index that captures per CMSA how many workers are available per job in an area. When many workers are available per job, wages and the cost of living are higher. Similarly, our model also predicts that rents are higher in good locations with many buyers per seller.

In most of this paper, the quality distribution is exogenous.[4] Menzio (2023) endogenizes the quality distribution by letting ex-ante homogeneous sellers choose their quality in a dynamic Burdett-Judd (1983) framework. He allows search frictions to decline over time. In response, firms offer more specialized products with a higher consumer value. In this environment, it is possible that, despite the decline in search frictions, the economy exhibits a balanced growth path where price dispersion and the extent of competition remain constant. This paper shares the observation that specialized sellers need traffic more than generic ones but it is complementary to Menzio (2023) in the sense that we study what happens when there is cross-sectional rather than time variation in search frictions. This adds a location choice and sorting dimension to the firm's problem. Albrecht et al. (2023) assume that firms' key design choice is vertical rather than horizontal (as it was in Menzio, 2023). In their model, high-quality sellers have higher trading probabilities because buyers visit multiple sellers and high-quality sellers can offer more surplus to buyers than low-quality sellers. In our model, buyers can only visit one seller but sellers can choose a good location to increase their trading probability at the cost of a higher rental price.

There exists a small literature that relates spatial sorting to search frictions. In Helsley and Strange (1990), match quality is higher in large urban areas because workers are more likely to find a job that matches their skills when there are many firms available. They also look at the optimal number of workers and firms to locate in a city subject to a population constraint and find that it is optimal to have cities of equal size. We find the same when products have the same quality. However, when (vertical) quality differs across sellers and when we allow for niche products, heterogeneous locations generate more surplus than identical areas.

[4]We do discuss how a fixed production cost interacts with the distribution of locations. That is, certain niche products will only be offered when location quality is sufficiently dispersed.

Gautier and Teulings (2009) assume a meeting function with increasing returns to scale, which makes large urban areas more efficient search markets. They find that workers with rare skills and firms that need to hire a wide variety of skills benefit most from dense labor markets. Combes et al. (2008) and Dauth et al. (2022) show that high-skilled workers sort into dense areas in respectively France and Germany. Kim (1989) does not have search frictions but in his model, workers do specialize more (rather than invest in general skills) in large markets because the fewer firm types there are, the less likely it is that one of them will demand a particular skill. In the context of the marriage market, Gautier et al. (2010) find that highly-educated singles (more so than couples or singles with less education) locate in big cities to find a partner because the opportunity cost of remaining unmatched are largest for them. Here, we have constant-returns-to-scale and many-on-one meetings. Sellers of high-quality and niche products create a lot of surplus conditional on matching with a buyer and they locate in areas where the probability that they meet multiple buyers is large. In marriage market terms, our model implies that niche types (types that belong to a niche subculture) sort into large cities.

Pissarides (2000) allows firms and workers to invest in search and recruitment intensity. This is modeled in a reduced form way as a scalar that increases the individual matching rate. At the aggregate level, if firms and workers double their search intensity, their meeting rates are also doubled. In our model, a seller can increase its meeting rate by moving to a good location where there are many buyers per seller but if some locations are better than average, it implies that other locations are worse than average (due to the buyer resource constraint). Since the price of a location is endogenous in our model, so is the price for a higher meeting rate whereas in reduced form models, the cost of increasing search intensity is exogenous.

In the urban economics literature, location choice is driven by the trade-off between positive agglomeration effects which give rise to increasing returns and mobility cost. The first is needed to explain why large cities exist and the second why not all jobs are in one large city.[5] This literature is mostly complementary to this paper. We have not much to say about the size distribution of cities. Our aim is to understand what types of sellers locate in attractive areas characterized by high buyer-seller ratios, why buyer-seller ratios differ across space, how this shapes the quality distribution of products that are offered and whether heterogeneity in location quality is desirable or not.

The paper is organized as follows. In section 2, we introduce the model, define equilibrium and characterize the model for a given location distribution. We analytically solve the model for two examples and then use a first-order approximation to study the effects of making

[5] See for example Ellison and Glaeser (1997), Fujita and Thisse (2002), Rosenthal and Strange (2004) Ellison et al. (2010), Moretti (2012).

goods more niche. Then, in section 3, we endogenize the distribution of locations and ask from an urban-planning point of view what the optimal distribution of locations is for a given quality distribution and what the market outcome would be if sellers form a coalition which chooses the seller-optimal distribution of locations, as in a real estate investment trust (REIT). Finally, section 4 concludes.

2 Exogenous Distribution of Locations

Suppose that that there are many chefs but only few of them have the talent to cook at the three-star level. At what locations would they open restaurants? What locations would less-talented chefs choose? And how would the planner assign chefs to locations? We address those questions using the following stylized model.

2.1 Environment

Agents. Consider a static environment with a measure 1 of sellers and a measure λ of buyers. Both buyers and sellers are risk neutral. Each seller possess a single unit of an indivisible good and each buyer has (inelastic) demand for one unit. Buyers and sellers who do not trade obtain a zero payoff.

Quality. Goods are characterized by their quality z and the corresponding probability $x(z)$ that a buyer likes the good. The distribution of z among sellers is given by a cumulative distribution $F(z)$. We assume that $x(z)$ is weakly decreasing in z, while $zx(z)$, the expected value of good z to a buyer, is weakly increasing in z. We can think of goods with a low value of x (or equivalently, a high value of z) as niche products.

Search. Search is random and occurs in space where we allow for heterogeneity in locations. Sellers in good locations meet relatively many buyers. We can think of the good locations as places that are easily reachable, for example because they are close to a station or a highway. Alternatively, they could be in a popular shopping street or mall in the city center. We model locations as points on a circle with circumference 1.

To better explain the search process, we first consider the case where the number of sellers (N_s) and buyers (N_b) are finite, which we illustrate in Figure 1 for $N_s = 6$. All sellers randomly arrive on the circle according to a probability distribution that we describe below. After that, buyers arrive on the circle and go clockwise to the nearest seller. We assume that buyers are placed uniformly on the circle, which, as we will see later, is a normalization rather than an assumption.

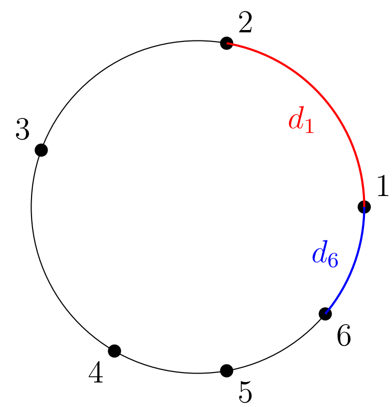

Figure 1: Finite number of sellers

From the perspective of seller i, the quality of their location depends on the arc distance to their counterclockwise neighbor, denoted by d_i. In Figure 1, d_i takes one of two values: half of all sellers have good spots and the other half have bad spots, where the arc length of a good spot is two times that of a bad spot. Hence, $d_1 = 2d_6 = 2/9$. Since each buyer visits seller i with probability d_i, the probability that seller i meets n buyers is given by

$$\binom{N_b}{n} d_i^n \left(1 - d_i\right)^{N_b - n}. \tag{1}$$

Below, we will allow for a continuum of buyers and sellers with $N_b \to \infty$ and $N_b/N_s \to \lambda$. We can write the expected buyer-seller ratio at location i as $N_b d_i \to \lambda s_i$ where $s_i = d_i N_s$. Equation (1) then converges to $e^{-\lambda s_i}(\lambda s_i)^n/n!$, that is, a Poisson distribution with mean λs_i. So in a large market, seller i's spot is characterized by s_i and we can think of good locations as locations where λs_i is large. The advantage of using s_i as a measure of location quality (for a given N_s) rather than d_i is that $\sum_{i=1}^{N_s} s_i/N_s = 1$, i.e., the mean of s_i is 1 by construction, while the mean of $d_i \to 0$ when we let the market get large. We denote the cumulative distribution function of location quality s by $L(s)$. The probability density for the event that a buyer meets a seller at a location s is then $s\,dL(s)$.[6]

We can relax the assumption that buyers arrive uniformly on the circle and assume that their arrival obeys some other probability distribution. However, this does not make the model more general, since it is $L(s)$, instead of the individual distributions of sellers and buyers, that matters for the market equilibrium.[7]

[6] In the example of Figure 1, the probability that a buyer meets a seller in location $s = 4/3$ is thus $4/3 \cdot 1/2 = 2/3$ and the corresponding probability for $s = 2/3$ is $2/3 \cdot 1/2 = 1/3$.

[7] Note that here we treat the distribution of locations, $L(s)$, as exogenous. In Section 3, we consider the case where the distribution of locations is endogenous.

Market for Locations. After the realization of their locations and before meeting buyers, sellers can trade their locations in a competitive, frictionless market. The price of location s is denoted by $r(s)$.

Surplus in a Location. After sellers select their preferred location and sell their old location, buyers and sellers meet. Suppose that a seller offers quality z and has queue length λs (because its location is s). The expected surplus for this seller is given by

$$S(s,z) = z \sum_{n=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n}{n!} \left[1 - (1 - x(z))^n\right] = z \left(1 - e^{-\lambda s x(z)}\right), \tag{2}$$

where the summand on the left-hand side denotes the scenario where the seller meets n buyers and at least one likes the product. Alternatively, since the *effective queue length* (the expected number of buyers who value the product) is $\lambda s x(z)$, the probability that the seller meets at least one buyer who likes the product is given by the term in parenthesis on the right-hand side.

Trade and Payoffs. Sellers select the buyer they trade with by means of a second-price auction (with a reserve price equal to the seller's valuation of zero). The expected payoff of a seller of type z at location s is then

$$\pi(s,z) = z \sum_{n=2}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n}{n!} \left[1 - (1-x)^n - nx\,(1-x)^{n-1}\right] = z \left(1 - e^{-\lambda s x} - \lambda s x e^{-\lambda s x}\right), \tag{3}$$

where we have suppressed the argument z from the function $x(z)$ to save space. The summand on the left-hand side denotes the probability that the seller meets n buyers and at least two of the n buyers value the good, in which case the transaction price equals z (if only one buyer arrives who likes the good, this buyer will just bid the reserve price). As in (2), since the effective queue length is $\lambda s x$, the probability that the seller meets two or more buyers who value the product is given by the term in the parenthesis on the right-hand side.

To simplify exposition later, we define

$$\mathcal{P}(\lambda) = 1 - e^{-\lambda} - \lambda e^{-\lambda} \tag{4}$$

which is the probability for the seller to be on the "short side" by meeting two or more buyers when the expected queue length is λ. Sellers' payoff in equation (3) can then be rewritten as $\pi(s,z) = z\mathcal{P}(\lambda s x)$.

The expected payoff of a buyer who meets this seller is

$$zx \sum_{n=0}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n}{n!} (1-x)^n = zxe^{-\lambda sx}, \tag{5}$$

where, as before, we have suppressed the argument z from the function $x(z)$. When a buyer meets a seller at location s, the buyer receives a strictly positive payoff if and only if the buyer likes the product, which happens with probability $x(z)$, and the seller meets no other buyers who like the seller's product. The summation on the left-hand side denotes the probability that the seller meets no other buyers who value the product, which, as argued above, is simply $e^{-\lambda sx}$.

Sellers' Location Choice. Before meeting the buyers, the problem of a seller of type z is to maximize the expected payoff from choosing location type s, which is given by

$$\max_s \pi(s,z) - r(s), \tag{6}$$

where $\pi(s,z)$ is given by equation (3). Equation (6) abstracts from the earnings associated with sellers selling their location endowment as that is irrelevant for the location choice problem.[8]

Rental Price. Suppose that in equilibrium $z^*(s)$ is the seller type that chooses location s.[9] Furthermore, suppose that the support of $L(s)$ is an interval. The first-order condition for the sellers' optimal choice of locations then implies that the gradient of the location prices equals

$$r'(s) = \pi_s(s, z^*(s)) = \lambda^2 s z^*(s) x^*(s)^2 e^{-\lambda s x^*(s)}, \tag{7}$$

where $x^*(s) \equiv x(z^*(s))$.

Equilibrium Definition. We can now define an equilibrium as follows.

[8] We have assumed that sellers are also owners of a random store/location. If we consider seller entry, this assumption may have the unrealistic implication that some low-quality sellers would enter just to sell their location endowment. Alternatively, we could assume that sellers and location owners are distinct groups, and there exists a large measure of potential sellers considering whether to enter the market. Then, there exists a threshold seller type above which sellers will enter the market (see Lemma 2 below for conditions under which $\pi(s,z)$ is strictly increasing in z). Given the set of active locations and sellers, both approaches result in the same equilibrium allocation between sellers and locations.

[9] To simplify exposition, we assume that there is a unique seller type $z^*(s)$ for each s, which holds for sure when the distribution of locations $L(s)$ is continuous without mass points. More generally when $L(s)$ contains mass points, the allocation between sellers and locations can be described by a joint distribution of (s,z) whose marginal distributions are $F(z)$ and $L(s)$.

Definition 1. *An equilibrium is an assignment of sellers to locations $z^*(s)$ and a price schedule for locations $r(s)$ such that*

1. *Seller optimality: Given $r(s)$, sellers' choices of locations maximize their expected profit. That is, each seller solves the problem given by (3).*

2. *The market for locations clears: The price schedule for locations is such that for each s, the demand for type-s locations equals their supply.*

2.2 Planner's Problem

The planner's problem is to match sellers with locations to maximize total net surplus. Recall that $S(s, z)$, the expected surplus between a seller z and a location s, is given by equation (2).

Lemma 1. *$S(s, z)$, which is defined by equation (2), is strictly increasing in z. Furthermore, it is strictly supermodular in (s, z) for any $\lambda > 0$.*

Proof. See Appendix A.1. □

Since surplus $S(s, z)$ is supermodular in (s, z), the planner's solution is characterized by PAM: better-quality sellers are assigned to better locations.[10] Suppose that both F and L are continuous distributions. The optimal assignment is then captured by the following correspondence,

$$1 - F(z) = 1 - L(s_p(z)), \tag{8}$$

where $s_p(z)$ is the optimal location s for seller type z.

Given the optimal matching between sellers and locations, expected total surplus is

$$Y(\lambda) = \int_s z_p(s) \left(1 - e^{-\lambda s x_p(s)}\right) dL(s), \tag{9}$$

$x_p(s) \equiv x(z_p(s))$ and $z_p(s)$ is the inverse of $s_p(z)$. Note that $Y(\lambda)$ is strictly concave in λ.

2.3 Market Equilibrium

Recall that the expected profit for a seller z from selling the good after choosing location s is $\pi(s, z)$, as given by equation (3). As is well known since Becker (1973), strict supermodularity

[10] The assumption that $x(z)$ is weakly decreasing is critical for the supermodularity of $S(s, z)$. Suppose to the contrary that $x(z)$ is increasing. Then, good locations (with large s) can be more valuable for low quality sellers because they need more buyer traffic to meet at least one buyer who likes their product. That is, s and z can be substitutes. For example, if $x(z) = z/(1+z)$, then $S_{sz}(s, z) > 0$ if and only if $\lambda s < (1+z)(2+z)/z$.

of $\pi(s, z)$ in (s, z) implies that the market outcome must be PAM: better-quality sellers choose better locations. However, the following lemma shows that unlike the surplus function $S(s, z)$, sellers' expected profit $\pi(s, z)$ is not necessarily supermodular, unless the elasticity of $x(z)$ is greater than $-1/2$.

Lemma 2. *Consider $\pi(s, z)$ defined by equation (3). Given z, if $\varepsilon_x(z) \equiv zx'(z)/x(z) \geq -1/2$, then the derivatives $\pi_z(s, z) > 0$ and $\pi_{sz}(s, z) > 0$ for any $\lambda > 0$.*

Alternatively, suppose that $\varepsilon_x(z) \in [-1, -1/2)$. Then

1) $\pi_{sz}(s, z) > 0$ if and only if the effective queue length $\lambda s x(z) > 2 + 1/\varepsilon_x(z)$.

2) $\pi_z(s, z) > 0$ if and only if

$$1 + \varepsilon_x(z)\varepsilon_{\mathcal{P}}\left(\lambda s x(z)\right) > 0 \tag{10}$$

where $\varepsilon_{\mathcal{P}}(\lambda) \equiv \lambda \mathcal{P}'(\lambda)/\mathcal{P}(\lambda)$, and $\mathcal{P}(\lambda)$ is defined by (4), which is the probability that a seller meets two or more buyers. Let $\Lambda(\cdot)$ be the inverse function of $\varepsilon_{\mathcal{P}}(\lambda)$. Then $\Lambda(\cdot)$ is strictly decreasing, and condition (10) holds if and only if $\lambda s x(z) > \Lambda(-1/\varepsilon_x(z))$.

3) If $\pi_z(s, z) > 0$, then $\pi_{sz}(s, z) > 0$. Equivalently, $\Lambda(-1/\varepsilon_x(z)) > 2 + 1/\varepsilon_x(z)$ for any $\varepsilon_x(z) \in [-1, -1/2)$.

Proof. See Appendix A.2. □

Supermodularity concerns how the marginal value of s varies with z. Thus when $zx(z)$ is increasing in z, then surplus $S(s, z)$ is supermodular. However, profit $\pi(s, z)$ is supermodular when $zx(z)^2$ (the probability that two effective buyers arrive is $x(z)^2$, ignoring higher order terms) is increasing in z, which is equivalent to $zx'(z)/x(z) \geq -1/2$. This condition on the elasticity says that if the chance that a buyer likes the product does not decrease too fast with the quality of the product, high-quality sellers benefit more from good locations than low-quality sellers. If the market for high-quality products becomes thin very fast (the elasticity $zx'(z)/x(z) \leq -1/2$) then even at good locations the probability to get two or more effective buyers is too small for the high-quality sellers to be willing to pay the higher rents there.

Since $\pi(s, z)$ can be written as $z\mathcal{P}(\lambda s x(z))$, it is increasing in z if and only if the elasticity of $\mathcal{P}(\lambda s x(z))$ with respect to z is greater than -1, i.e., condition (10) holds. Given that the elasticity of $\mathcal{P}(\cdot)$ is strictly decreasing, $\pi(s, z)$ is increasing in z if and only if the effective queue length $\lambda s x(z)$ is large enough so that the decrease in $\mathcal{P}(\lambda s x(z))$ is moderate. Furthermore, the threshold above which $\pi(s, z)$ is increasing in z is always greater than the threshold for supermodularity; thus, when $\pi(s, z)$ is increasing in z, $\pi(s, z)$ is always supermodular.

When $\pi(s,z)$ is supermodular, the quality of sellers who choose location s in equilibrium is also given by equation (8), as in the planner's solution. We thus have the following proposition.

Proposition 1. *Suppose that $zx'(z)/x(z) \geq -1/2$ for any z. The decentralized equilibrium is then constrained efficient.*

When $zx'(z)/x(z) < -1/2$, it is possible that the equilibrium matching between sellers and locations does not exhibit PAM and hence the decentralized equilibrium is not efficient. To see this, note that a necessary condition for PAM to hold is that along the equilibrium path, for each s the cross-partial derivative $\pi_{sz}(s, z^*(s)) \geq 0$, where $z^*(s)$ is given by equation (8) and is independent of λ. If the lower support of $L(s)$ is zero, then for small s the effective queue length $\lambda x(z)s$ is close to zero, and by Lemma 2, $\pi_{sz}(s, z^*(s))$ is strictly negative. Hence, the necessary condition does not hold.

The reason that the planner's solution can differ from the market outcome is that in the planner's solution, a higher s increases surplus $S(s,z)$ because it increases the probability that a seller meets at least one effective buyer, whereas in the competitive market of locations, it increases $\pi(s,z)$ (the gross payoff of buyers of locations) because it increases the probability that a seller meets at least two effective buyers who then bid the price up. The investment of a seller in a good location is therefore partly a rent-seeking activity because the returns of receiving more than one buyer are zero from a social welfare point of view but positive from an individual firm's point of view.

Specifically, recall that given z, $\pi(s,z)$ as a function of s has an S-shape: $\mathcal{P}(\lambda x(z)s)$ is first convex and then concave in s. In the convex range of $\mathcal{P}(\cdot)$ where the buyer-seller ratio is low, sellers of general products benefit more from good locations than sellers of high quality, niche products because the latter group rarely meets two or more buyers who like their product. To see this, consider two sellers $z^a < z^b$ who match with locations $s^a < s^b$, respectively. If the effective queue lengths for the two sellers are low, then assigning location s^b to seller z^a and location s^a to seller z^b will increase the total profits of the two sellers by exploiting the initial convexity of $\mathcal{P}(\cdot)$ but this will decrease social surplus. The *increase* in seller a's probability that two or more buyers visit at the good location exceeds the *decrease* in seller b's probability that two or more buyers visit at the bad location: $\mathcal{P}(\lambda x(z^a)s^b) - \mathcal{P}(\lambda x(z^a)s^a) > \mathcal{P}(\lambda x(z^b)s^b) - \mathcal{P}(\lambda x(z^b)s^a)$, or equivalently $\int_{s^a}^{s^b} \mathcal{P}'(\lambda x(z^a)s)ds > \int_{s^a}^{s^b} \mathcal{P}'(\lambda x(z^b)s)ds$, where we used the fact that $x(z^a) > x(z^b)$ and $\mathcal{P}(\widetilde{\lambda})$ is convex when $\widetilde{\lambda}$ is small. Furthermore, the increase in seller a's probability that two or more buyers visit can be sufficiently large such that it outweights z^b/z^a times the corresponding decrease of seller b's probability.

The inefficiency discussed above can justify city planning policies that prioritize or discourage certain shops in specific neighborhoods, using tools such as rental subsidies or taxes.

However, implementing these policies requires comprehensive information about buyer preferences, the distribution of buyers, and the distribution of locations. Recall that the planner's optimal location s for seller type z is $s_p(z)$, as given in equation (8). The effective queue length, $\lambda x(z)s_p(z)$, can be increasing, decreasing, or non-monotonic in z; its monotonicity depends on both $x(z)$ (buyer preferences) and $s_p(z)$, where the latter is jointly determined by the distributions of seller types and locations. If the decrease in $x(z)$ outweighs the increase in $s_p(z)$, then $\lambda x(z)s_p(z)$ decreases in z. This implies that sellers with high z become targets for policy interventions, as the supermodularity of $\pi(s_p(z), z)$ fails when the effective queue length $\lambda x(z)s_p(z)$ is too small. Conversely, if $\lambda x(z)s_p(z)$ increases in z, then sellers with low z become the focus of policy interventions.

2.4 Two Examples

Below, we analytically characterize the planner's solution for some parametric examples of seller-type and location-quality distributions.

2.4.1 Vertical Quality ($x(z) = 1$ for any z)

In our first example, buyers like each product with certainty, $x(z) = 1$ for all z. As a result, all buyers agree on the ranking of products, such that quality differences between sellers can be viewed as being vertical. We assume that the quality distribution follows a power law and that the location distribution is exponential. Specifically, let $F(z) = 1 - \left(\frac{z_0}{z}\right)^\alpha$ with $\alpha > 1$ and $z \geq z_0 > 0$ and let $L(s) = 1 - e^{-\rho(s-s_0)}$ where $s \geq s_0$, $s_0 \in [0, 1)$ and $\rho = (1 - s_0)^{-1}$.

Expected Total Surplus. Since $x(z) = 1$ for all z, the equilibrium in the market for locations exhibits PAM and thus coincides with the planner's solution (see Lemma 1 and 2). By equation (8), the correspondence between sellers and locations is

$$e^{-(s-s_0)/(1-s_0)} = \left(\frac{z_0}{z}\right)^\alpha,$$

which implies that

$$z^*(s) = z_0 e^{(s-s_0)/(\alpha(1-s_0))}. \tag{11}$$

Substituting the above equation into equation (9) implies that total surplus equals

$$Y(\lambda, \alpha, s_0) = \int_s z^*(s)\left(1 - e^{-\lambda s x^*(s)}\right) dL(s) = \overline{z}\left(1 - \frac{(\alpha-1)e^{-\lambda s_0}}{(\alpha-1) + \alpha\lambda(1-s_0)}\right), \tag{12}$$

where $\overline{z} = z_0\alpha/(\alpha - 1)$ denotes the mean of z and we have added (α, s_0) as arguments of $Y(\lambda)$ to show that total surplus depends on quality and location dispersion. The following lemma characterizes this dependence.

Lemma 3. *If we increase both α and z_0 so that the average quality $\overline{z} = z_0\alpha/(\alpha - 1)$ is constant, then $Y(\lambda, \alpha, s_0)$ decreases.*

If $\alpha\lambda \leq 1$, $Y(\lambda, \alpha, s_0)$ is strictly decreasing in $s_0 \in [0, 1]$. If $\alpha\lambda > 1$, then $Y(\lambda, \alpha, s_0)$ is first increasing and then decreasing in s_0, reaching a maximum at $s_0 = 1 - 1/(\alpha\lambda)$.

Proof. See Appendix A.3. □

Increasing α, while adjusting z_0 to keep the mean $\overline{z}$ fixed, reduces quality dispersion. In the limit $\alpha \to \infty$, all sellers have the same quality z. Hence, the fact that locations are heterogeneous makes quality dispersion desirable because of the complementarity between quality and location.

The effect of location dispersion on total surplus is more complicated. All locations are the same when $s_0 = 1$, while location dispersion is maximized when $s_0 = 0$. When λ is small ($\leq 1/\alpha$), the increase in the matching probability of high-quality sellers dominates the overall decrease in matches so location dispersion is desirable. When λ is large, too much location dispersion leads to a minor increase in the matching probability of high-quality sellers while it reduces the matching probability of low-quality sellers substantially and location dispersion is not desirable.

Sellers' Profit. The expected payoff of sellers before entering the market is given by

$$\Pi(z) = \pi(s^*(z), z) - r(s^*(z)) + R, \tag{13}$$

where $s^*(z)$ is the sellers' optimal location choice, and $R = \int_s r(s)\, dL(s)$ is the average location price, which is also the seller's expected payoff from selling their location since they are randomly endowed with a location according to the distribution $L(s)$.

To analytically solve for the ex-ante payoff of a seller, we consider the two extreme cases $s_0 = 0$ and $s_0 = 1$. We will use subscripts 0 and 1 to compare variables between the two cases. Assume that $\alpha\lambda = 1$ (this yields a particular simple expression for $s_0 = 0$).

When $s_0 = 1$, all locations are the same. Sellers will not trade locations and their expected profit is $\Pi_1(z) = z(1 - e^{-\lambda} - \lambda e^{-\lambda})$. The result for the case $s_0 = 0$ is given by the following lemma.[11]

[11] When comparing the two cases in Proposition 2, we fix the measure of buyers. We can also show that a similar result holds when we assume free entry of buyers, i.e., buyers have to incur cost K to enter the market. Thus, the measure of buyers will differ between the two cases. The proof is available upon request.

Proposition 2. *Consider two location distributions:* $L_0(s) = 1 - e^{-s}$ *and* $L_1(s)$ *is degenerate at* $s = 1$*. Suppose that* $x(z) = 1$ *for all* z*, and* $F(z) = 1 - (z_0/z)^\alpha$ *with* $\alpha > 1$*. Let* $\Pi_i(z)$ *be the ex-ante seller payoff under location distribution* $L_i(s)$ *with* $i = 0, 1$*, which is defined by equation* (13)*. Furthermore, assume that* $\lambda\alpha = 1$*.*

Then $\Pi_0(z)$ *is strictly convex with* $\lim_{z\to\infty} \Pi_0'(z) = 1$*, and* $\Pi_0(z) > \Pi_1(z)$ *for all* $z \geq z_0$*.*

Proof. See Appendix A.4. □

Two things stand out when comparing sellers' expected payoffs under the two location distributions (heterogeneous locations with $s_0 = 0$ and identical locations with $s_0 = s = 1$). First, when $s_0 = 0$, the sellers with the worst type z_0 have $s = 0$ in equilibrium (they meet no buyers so this is equivalent to being inactive). When $s_0 = s = 1$, each seller expects an effective queue length λ. One may expect that sellers with low z are worse off under $s_0 = 0$ than under $s_0 = 1$. However, the above proposition shows that when $\alpha\lambda = 1$, the expected payoff from selling one's location endowment is so high that all sellers are better off. When the location distribution is dispersed, sellers with high z are willing to pay a high price for good locations. This not only benefits high-quality sellers but can also benefit low-quality sellers, since those sellers are also location owners.

Second, so far we have assumed that sellers incur no costs of participating in this market. Suppose now that sellers need to pay a production cost $c(z)$ to enter the market, which is assumed to be weakly increasing and convex. Suppose that $c'(z) > 1 - e^{-\lambda} - \lambda e^{-\lambda}$ for large z. When all locations are the same ($s_0 = 1$), it will then not be profitable for high-quality sellers to enter the market. If $c'(z) < 1$ for all z, it is possible that all sellers, including the high-quality niche ones, find it profitable to enter the market when locations are hetergeneous ($s_0 = 0$). In the latter case, the good locations allow the high-quality sellers to meet many buyers (and thus trade and create surplus with high probability), whereas offering a high-quality product is too risky when all locations are identical, because there is a substantial chance that too few buyers arrive.

2.4.2 Power Law for Both Distributions ($F(z)$ and $L(s)$)

In our second example, we allow for heterogeneity in $x(z)$. Neiman and Vavra (2023) document that products that are more niche are offered in large dense cities. That is, niche sellers with a smaller x and larger z will choose a larger s. However, it is not clear whether the effective queue length λxs (or equivalently sellers' trading probability) is increasing or decreasing in seller types. To show that both options are feasible, we construct a knife-edge example where the effective queue length (or sellers' trading probability) remains constant. By perturbating this example, a seller's trading probability can either increase or decreases with z (see our third example below for a first-order approximation approach).

We assume that both the quality distribution and location distribution follow power laws. Specifically, let $F(z) = 1 - \left(\frac{z_0}{z}\right)^\alpha$ with $\alpha > 1$ and $z \geq z_0$ and let $L(s) = 1 - \left(\frac{s_0}{s}\right)^\beta$ where $s \geq s_0$ and $\beta > 1$. Since the mean of s must be 1, we have $s_0 = (\beta - 1)/\beta \in (0, 1)$. Furthermore, let $x(z) = \left(\frac{z_0}{z}\right)^\gamma$ where $0 < \gamma \leq 1$. A smaller γ implies that the probability that a buyer likes the good declines less quickly in z, bringing us closer to the case of vertical quality. Note that $\mathbb{E}z = \overline{z} = z_0\alpha/(\alpha - 1)$ and $\mathbb{E}zx(z) = z_0\alpha/(\alpha - 1 + \gamma)$. We then consider the knife-edge case where $\alpha = \beta\gamma$ or equivalently $\alpha(1 - s_0) = \gamma$, since it yields analytical tractability.

Given that the assignment between sellers and locations is PAM at the planner's solution, the correspondence between sellers and locations is

$$\left(\frac{z_0}{z}\right)^\alpha = \left(\frac{s_0}{s}\right)^\beta$$

which implies that

$$z^*(s) = z_0 \left(\frac{s}{s_0}\right)^{\beta/\alpha}. \tag{14}$$

The matching probability for a seller of type z is given by $1 - e^{-\lambda x(z) s^*(z)} = 1 - e^{-\lambda s_0}$, which is constant across different sellers. Total expected surplus is then given by

$$Y(\lambda, \alpha, s_0) = \left(1 - e^{-\lambda s_0}\right) \overline{z}. \tag{15}$$

Since all sellers have the same trading probability, the expected total surplus is simply the matching probability times the average quality.

2.5 Comparative Statics: Making Goods More Niche

Qualitative Effects. A simple consequence of the above framework is that making the goods simultaneously better and more niche such that the expected buyer value remains the same (increase z and decrease x simultaneously so that $zx(z)$ is constant), increases total surplus for each λ. The intuition is that since there are no meeting externalities among buyers, making the goods more niche takes advantage of the fact that sellers can meet multiple buyers simultaneously and they only need one buyer who likes their product (recall that with niche products, those who like it, like it a lot). In other words, if we reduce $x(z)$ by half, then the probability that at least one buyer likes the product is reduced by less than half, since now it is less likely that multiple buyers like the product simultaneously, so that sellers can take advantage of meeting multiple buyers more effectively.[12]

[12] To see this, consider the scenario that a seller meets two buyers. The probability that at least one of the two buyers likes the product is then $2x(z) - x(z)^2$. If we reduce $x(z)$ by half, then the probability that

The case of homogeneous sellers is particularly simple: The effect of a percentage increase in z while holding $zx(z)$ constant is equivalent to a percentage increase in the measure of sellers, which can be easily seen from the simplest case where all locations are the same and total surplus can be written as $N_s z(1 - e^{-N_b x(z)/N_s})$. The same logic applies to the case of heterogeneous locations.

When sellers are heterogenous, making the goods more niche while fixing the distribution of $q = zx(z)$ again increases total surplus. If the percentage increase of z is the same for different values of z, then this is equivalent to a corresponding percentage increase in the measure of sellers. More generally, it matters how the percentage increase of z depends on the value of z. For example, if we increase z to z^2 while reduce $x(z)$ from 1 to $1/z$ for $z \geq 1$, then the expected quality stays the same, and the goods becomes more niche. The percentage increase of z, i.e., z^2/z, is higher for larger z. Below we use a first-order approximation approach to analyze how this type of change affects total surplus.

Quantitative Effects: A First-Order Approximation Approach. We now set $x(z) = \left(\frac{z_0}{z}\right)^\gamma$ as in the previous two examples, and analyze how an increase of γ, while holding the distribution of $zx(z)$ constant, increases total surplus. In this case, the percentage increase in z (due to an increase in γ) is higher for a higher fixed expected value $q = zx(z)$. To see this, (with a slight abuse of notation) let $z(q,\gamma)$ and $x(q,\gamma)$ be the value and trading probability associated with q and γ. Since $zx(z) = q$ and $x(z) = \left(\frac{z_0}{z}\right)^\gamma$, we have $z(q,\gamma) = z_0(q/z_0)^{1/(1-\gamma)}$ and $x(q,\gamma) = (q/z_0)^{-\gamma/(1-\gamma)}$. The effect of a higher γ on the percentage change of $z(q,\gamma)$ is

$$\frac{\partial \log z(q,\gamma)}{\partial \gamma} = \frac{\log(q/z_0)}{(1-\gamma)^2},$$

which is strictly increasing in q. Thus, the percentage increase in $z(q,\gamma)$ (due to an increase in γ) is higher for a higher fixed expected value q.

In the above two examples, γ equals 0 or $\alpha(1-s_0)$, respectively; an analytic expression for total surplus for general γ is difficult to obtain. We thus adopt a first-order approximation approach for the examples above to study the effects of increasing γ on surplus.[13] Furthermore, we analyze how this effect depends on α, which measures the dispersion of seller quality. Recall that in the two examples above, we assumed that the distribution of z follows a power law, i.e. $F(z) = 1 - \left(\frac{z_0}{z}\right)^\alpha$ with $\alpha > 1$ and $z \geq z_0$. In Online Appendix B.1, we show that in both examples, the effect is smaller when α is larger, i.e., when quality dispersion is smaller.

both buyers like the product decreases by more than half. More generally, suppose that a seller meets n buyers where $n \geq 2$, and we reduce $x(z)$ to $\delta x(z)$ for $\delta \in (0,1)$. Then $1-(1-\delta x(z))^n > \delta(1-(1-x(z))^n)$. This inequality holds because the left-hand side is strictly concave in δ, and when $\delta = 0$ or 1, it equals the right-hand side, which implies that when $\delta \in (0,1)$, it is strictly greater than the right-hand side.

[13] Let γ be slightly above 0 and $\alpha(1-s_0)$, respectively, for these examples, while holding other factors fixed.

2.6 Price Posting

In real world markets, products are often sold by price posting. In this section, we show that our main conclusion (the decentralized equilibrium is sometimes efficient and sometimes not) still holds under price posting, and that the insights obtained from our benchmark model under what conditions spatial sorting can be inefficient also carry over.

A simple and straightforward way to model price posting is to make buyers ex-post heterogeneous. Suppose that after entering the market and meeting a seller of quality z, buyers draw a valuation $\widetilde{z}$ from a distribution $G(\widetilde{z}, z)$.[14] Sellers with one or more visiting buyers who are willing to pay p can pick one buyer and charge the posted price (so meetings are effectively random). Then, the problem of an (s, z) seller is to choose p to maximize expected profit:

$$\pi(s,z) \equiv \max_p p \sum_{n=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n}{n!} \left(1 - G\left(p, z\right)^n\right) = \max_p p(1 - e^{-\lambda s(1-G(p,z))}).$$

In Online Appendix B.2, we show that for some distributions, $\pi(s, z)$ is always supermodular while for others it is not and inefficient spatial sorting can occur. For example, if $G(p, z) = H(p/z)$ where $H(\cdot)$ is a univariate cdf and sellers' type z acts as a shifter for the distribution of buyers' values, $\pi(s, z)$ is always supermodular. For this class of distributions goods become niche not too fast as we increase quality.

Alternatively, as in our benchmark model, when goods become niche very fast as we increase quality, $\pi(s, z)$ can be submodular and the high-quality sellers no longer sort in the best locations. We illustrate this case with uniform distributions on the domain $[1-z, 1+z]$ which always have a mean equal to 1. When $z \to 0$, the distribution is degenerate at 1, while when $z \to 1$, the distribution has the maximal dispersion. That is, as z increases, the product becomes more niche. In Online Appendix B.2, we show that in general, when λs is small, $\pi(s, z)$ is not supermodular so spatial sorting can be inefficient. This is similar to the results from our benchmark model where sellers post auctions; see Lemma 2 on the case where $\varepsilon_x(z) \in [-1, -1/2)$. A complication here is that whether PAM (of sellers and locations) is socially desirable or not now depends on $G(\widetilde{z}, z)$. The example of inefficient sorting in Online Appendix B.2 is for a class of $G(\widetilde{z}, z)$ where it is socially desirable that the best sellers sort into the best locations.

[14]In our benchmark model, $G(\widetilde{z}, z)$ is a two-point distribution where $\widetilde{z} = z$ with probability $x(z)$ and 0 with probability $1 - x(z)$. Note that if $G(\widetilde{z}, z)$ is decreasing in z, then the buyer's value for high-quality goods first-order stochastically dominates that of low-quality goods.

2.7 Meeting Probabilities, Mixed Poisson and Invariance

While meeting data is widely available for both labor and product markets, we often do not observe location quality. In that case, meetings in the data will follow a mixed Poisson distribution where the probability that a seller meets n buyers is

$$P_n(\lambda) \equiv \int_s e^{-\lambda s} \frac{(\lambda s)^n}{n!} \, dL(s), \tag{16}$$

where $n \geq 0$.[15] Of particular importance is the probability that a seller meets at least one buyer:

$$m(\lambda) \equiv 1 - P_0(\lambda) = \int_s \left(1 - e^{-\lambda s}\right) dL(s). \tag{17}$$

If we know $m(\lambda)$, then $P_n(\lambda)$ can be derived as $P_n(\lambda) = (-1)^{n-1}\lambda^n m^{(n)}(\lambda)/n!$ for $n \geq 1$.

Below we give some examples of $L(s)$ where we can explicitly calculate $m(\lambda)$. Recall that the mean of s is always 1. The examples illustrate that many meeting distributions like the geometric and the gamma distribution can be thought of as spatial mixtures of Poisson distributions.

Example 1. If the distribution of s is degenerate at 1, then $m(\lambda) = 1 - e^{-\lambda}$ which is the standard urn-ball matching function.

Example 2. If $L(s)$ is an exponential distribution with $L(s) = 1 - e^{-s}$ where $s \geq 0$, then

$$m(\lambda) = \int_0^\infty \left(1 - e^{-\lambda s}\right) dL(s) = 1 - \frac{1}{1+\lambda}, \tag{18}$$

which is the geometric matching function (see e.g. Lester et al., 2015; Cai et al., 2017). More generally, let $L(s)$ be an exponential distribution with support $[s_0, \infty)$ where $s_0 \in [0, 1)$. That is, $L(s) = 1 - e^{-\rho(s-s_0)}$, where $\rho = (1 - s_0)^{-1}$ such that the mean of s is 1. Then

$$m(\lambda) = 1 - \frac{e^{-\lambda s_0}}{1 + \lambda(1 - s_0)}. \tag{19}$$

This matching function is strictly increasing in s_0 (for fixed λ). When $s_0 \to 1$, s gets more and more concentrated around 1 and $m(\lambda) \to 1 - e^{-\lambda}$.

Example 3. If $L(s)$ is a Gamma distribution with density $L'(s) = \rho^\rho s^{\rho-1} e^{-\rho s}/\Gamma(\rho)$ where

[15] When there is no market for locations and sellers are allocated randomly to spots, the meeting technology for a seller is also given by (16).

$s \geq 0$ and $\Gamma(\cdot)$ is the standard Gamma function, then

$$m(\lambda) = 1 - \left(\frac{\rho}{\rho + \lambda}\right)^{\rho}.$$

When $\rho = 1$, we have the geometric meeting technology, and when $\rho = \infty$, we have the urn-ball meeting technology. For $\rho \in (1, \infty)$, the corresponding $P_n(\lambda)$, defined in equation (16), follows a negative binomial distribution. This distribution is of interest since Davis and de la Parra (2017) provide empirical evidence that the number of job applications that vacancies receive can be well approximated by a negative binomial distribution, adjusted so that zero has a larger weight.

Relation with Invariant Meeting Technologies. In our setup, each buyer values the seller's product with probability $x = x(z)$. Hence, a seller only trades if this seller meets at least one such buyer, which happens with probability

$$1 - \sum_{n=0}^{\infty} P_n(\lambda)(1-x)^n = m(\lambda x), \tag{20}$$

where the summand on the left-hand side represents the scenario where the seller meets n buyers who all do not value the product.[16] The resulting meeting probability is $m(\lambda x)$, which depends only on the effective queue length λx that the seller faces before the location realization. That is, the meeting process between the seller and buyers who like the seller's product is not affected by how many buyers there are who do not like the product. Intuitively, the rate at which a vegan restaurant meets customers does not depend on how many people pass the restaurant who do not like vegan food. This property is called *invariannce*, see Lester et al. (2015) and Cai et al. (2017). In Online Appendix B.3, we show that all invariant meeting technologies can be written as mixtures of Poissons, and the expected payoff of buyers and sellers can be easily represented by function $m(\cdot)$ and its derivative. This representation of invariant meeting technologies is useful in other settings as well. For example, Becker and Mangin (2023) use it to study extreme outcomes in markets with search frictions.

3 Endogenous Distribution of Locations

We now consider the case where the distribution of locations is endogenous. That is, in the previous section, we allowed seller types to endogenously sort into different locations but the locations themselves were taken as given. Now, given the distribution of seller types, we ask what the most efficient distribution of locations is from a social point of view

[16] A detailed derivation of equation (20) can be found in Appendix A.5.

and whether this coincides with the market equilibrium where sellers form a coalition, and the coalition chooses a distribution of locations to maximize expected total seller profit. Since the market for locations is competitive, the equilibrium assignment between sellers and locations always maximizes the total expected seller profit, given the distribution of sellers and locations. Thus, the equilibrium with an exogenous distribution of locations is equivalent to the equilibrium that occurs when a coalition of sellers assigns sellers to locations in order to maximize total expected seller profit. In contrast, here, the coalition of sellers chooses *both* the distribution of locations *and* the assignment of sellers to locations in order to maximize total expected seller profit.

3.1 Planner's Problem

The planner's objective is to choose a distribution of locations $L(s)$ to maximize total net surplus, given the measure of buyers. Equivalently, we can assume that the planner allocates a location space $s(z)$ for each seller type, subject to the constraint that the total space is 1, i.e., the circumference of the circle.[17]

The planner's problem is thus given by

$$\max_{s(z)} \quad Y = \int_{z_0}^{\infty} z\left(1 - e^{-\lambda s(z) x(z)}\right) dF(z), \tag{21}$$

$$\text{s.t.} \quad \int_{z_0}^{\infty} s(z)\, dF(z) = 1. \tag{22}$$

As in equation (2), the integrand in the first line is the surplus generated by a seller of type z. Total expected surplus Y then follows by integrating across different seller types.

The Lagrangian of this problem is

$$\mathcal{L} = \int_{z_0}^{\infty} z\left(1 - e^{-\lambda s(z) x(z)}\right) dF(z) + \xi\lambda\left(1 - \int_{z_0}^{\infty} s(z)\, dF(z)\right),$$

where ξ is the modified multiplier (multiplied by λ). Let the planner's solution be $s_p(z)$. For each z, it must satisfy the first-order condition

$$z x(z) e^{-\lambda x(z) s_p(z)} = \xi \quad \text{if} \quad \xi < z x(z), \tag{23}$$

with complementary slackness (if $zx(z) \leq \xi$, then $s_p(z) = 0$). Since the objective function is strictly concave in $s(z)$ and the constraint is linear in $s(z)$, the FOC is both necessary and

[17]The planner chooses the same s for the same type of sellers (urn-ball is optimal when sellers are homogeneous).

sufficient for the planner's solution. Solving the FOC yields that for $zx(z) \geq \xi$,

$$s_p(z) = \frac{1}{\lambda x(z)} \left(\log zx(z) - \log \xi \right). \tag{24}$$

Since $zx(z)$ is increasing and $x(z)$ is decreasing, it follows that $s_p(z)$ is increasing. In other words, PAM between sellers and locations continues to hold.

Interestingly, the planner's solution is equivalent to the outcome in a directed search equilibrium (buyers observe the sellers' locations and the terms of trade ex ante), where the Lagrangian multiplier ξ corresponds to the market utility of buyers.

3.2 Market Equilibrium

To make the equilibrium distribution of locations endogenous, we assume that sellers form a coalition, as in a real estate investment trust (REIT). We consider the following two-stage process. First, the coalition chooses the distribution of locations. Second, given the distribution of locations, the market operates as in our benchmark model. That is, sellers are initially endowed with some location. Then sellers trade their locations in a competitive market. Finally, after sellers are matched to their locations, they randomly meet with buyers and each product is sold by a second-price auction. The objective of the coalition of sellers is to maximize the expected total seller profit.

Since sellers also own the locations, the equilibrium assignment of sellers to locations in the market for locations maximizes the expected total seller profit from selling to the buyers (by the first welfare theorem).Thus it is without loss of generality to assume that—instead of the two-stage process described above—the coalition chooses an arc length for each seller directly to maximize the expected total seller profit, which will be the approach that we follow below.

Homogeneous Sellers. To understand the coalition's problem, we first consider the special case where all sellers are homogeneous. As we illustrate below, since the maximization problem is not concave, we allow for the possibility that identical sellers have different s. Suppose that the coalition assigns s_i to a fraction of ℓ_i sellers where $i = 1, \ldots, I$, then the coalition's problem is

$$\max_{(s_i, \ell_i)} \quad \sum_{i=1}^{I} \ell_i \cdot z\mathcal{P}(\lambda x(z) s_i)$$

$$\text{s.t.} \quad \sum_{i=1}^{I} \ell_i s_i = 1.$$

Recall that $\mathcal{P}(\lambda)$ denotes the probability that seller z meets two or more effective buyers, as defined in equation (4). Since $\mathcal{P}(\lambda)$ is non-concave, the above problem is to maximize a convex combination of a non-concave function under a linear constraint, which requires finding the concave hull of $\mathcal{P}(\lambda)$ (i.e., the least concave function that is greater than it).

Since $\mathcal{P}(\lambda)$ has an S-shape, its derivative is first increasing and then decreasing. So, its concave hull is given by

$$\widehat{\mathcal{P}}(\lambda) = \begin{cases} \frac{\lambda}{\Lambda_1}\mathcal{P}(\Lambda_1) & \text{if } \lambda \leq \Lambda_1 \\ \mathcal{P}(\lambda) & \text{if } \lambda \geq \Lambda_1. \end{cases} \tag{25}$$

That is, when $\lambda \leq \Lambda_1$, $\widehat{\mathcal{P}}(\lambda)$ is the line segment between point $(0,0)$ and $(\Lambda_1, \mathcal{P}(\Lambda_1))$, and when $\lambda \geq \Lambda_1$, $\widehat{\mathcal{P}}(\lambda)$ coincides with $\mathcal{P}(\lambda)$. The threshold Λ_1 is determined by the condition that at $\lambda = \Lambda_1$, the slope of $\mathcal{P}(\lambda)$ equals the slope of the line segment, which implies that $\widehat{\mathcal{P}}(\lambda)$ has a continuous, decreasing derivative. Formally, Λ_1 solves $\mathcal{P}(\Lambda_1)/\Lambda_1 = \mathcal{P}'(\Lambda_1)$ or equivalently $\varepsilon_{\mathcal{P}}(\Lambda_1) = 1$, which is exactly $\Lambda(1)$ defined in Lemma 2 (see equation (10)) and its value is close to 1.8.

The coalition's solution for the case of homogeneous sellers is then straightforward. If $\lambda x(z) \geq \Lambda_1$, then all sellers will have the same s. In contrast, if $\lambda x(z) < \Lambda_1$, then the coalition will allocate a fraction of ℓ sellers to $s_1 = 0$ (equivalent to excluding those sellers from participation) and a fraction $1-\ell$ sellers to $s_2 = \Lambda_1/(\lambda x(z))$ where ℓ is such that $(1-\ell)s_2 = 1$. Therefore, the coalition's solution coincides with the planner's solution from the last subsection if and only if $\lambda x(z) \geq \Lambda_1$. In other words, the coalition's solution can be inefficient because i) in the planner's problem, the probability $1-e^{-\lambda}$ is strictly concave, and ii) in the coalition's problem, the probability $\mathcal{P}(\lambda) = 1-e^{-\lambda} - \lambda e^{-\lambda}$ is first convex and then concave.[18]

Heterogeneous Sellers. We now consider the general case where sellers are heterogeneous. To simplify exposition, we assume that $F(z)$ is continuous. Suppose that the coalition optimally assigns arc length $s_c(z)$ to a seller of type z. The above result shows that for any active seller, the coalition will set the effective queue length $\lambda x(z)s_c(z) \geq \Lambda_1$. Furthermore, by Lemma 2, $\pi(s,z)$ is strictly increasing in z since the effective queue length is always at least $\Lambda_1 = \Lambda(1)$, which implies that the coalition chooses a threshold $\widehat{z}$ below which $s(z)$ equals to zero. Of course, the coalition can set $\widehat{z} = z_0$, the lowest seller type, so that all sellers are active in this case.

[18]The property that $\mathcal{P}(\lambda)$, the probability that a seller meets two or more buyers, is first convex and then concave extends beyond the Poisson distribution. For example, in the class of negative binomial distributions, which includes the Poisson and geometric distributions as special cases, $\mathcal{P}(\lambda)$ is convex when $\lambda < \rho/(1+\rho)$ and concave otherwise.

Given that the effective queue length is always greater than $\Lambda_1 = \Lambda(1)$ so that $\pi(s,z)$ is strictly increasing in z, we have $\pi_{sz}(s^c(z),z) > 0$ by Lemma 2. Thus at the coalition's solution, PAM between sellers and locations always holds. That is, $s_c(z)$ is increasing in z. So, it follows immediately that the efficiency properties of the model with endogenous locations are quite different from the case where the location distribution $L(s)$ is exogenous and PAM between locations and sellers fails when the effective queue length for some sellers is small.

Formally, the coalition chooses $s(z)$ for each seller type z to maximize total surplus:

$$\max_{s(z)} \quad \int_z z\widehat{\mathcal{P}}(\lambda x(z)s(z))dF(z) \tag{26}$$

subject to the constraint (22), where $\widehat{\mathcal{P}}(\cdot)$ is given by equation (25). The Lagrangian of this problem is,

$$\mathcal{L} = \int_z z\widehat{\mathcal{P}}(\lambda x(z)s(z))dF(z) + \zeta\lambda\left(1 - \int_z s(z)dF(z)\right)$$

where ζ is the modified multiplier (multiplied by λ). For each z, the optimal solution $s_c(z)$ must satisfy the following first-order condition,

$$zx(z)\widehat{\mathcal{P}}'(\lambda x(z)s_c(z)) = \zeta \quad \text{if} \quad \zeta \leq zx(z)\mathcal{P}'(\Lambda_1) \tag{27}$$

with complementary slackness (if $zx(z)\mathcal{P}'(\Lambda_1) < \zeta$, then $s_c(z) = 0$). Note that the above first-order condition is both necessary and sufficient for the coalition's solution.

Since $\widehat{\mathcal{P}}(\lambda)$ is first linear and then strictly concave, and $zx(z)$ is (weakly increasing), the coalition's solution is characterized by a threshold $\widehat{z}$ below which $s_c(z) = 0$. If $\zeta \leq z_0x(z_0)\mathcal{P}'(\Lambda_1)$, then $\widehat{z} = z_0$; if $\zeta > z_0x(z_0)\mathcal{P}'(\Lambda_1)$, then $\widehat{z}$ is determined by the condition $\zeta = \widehat{z}x(\widehat{z})\mathcal{P}'(\Lambda_1)$ (in this case $\lambda x(\widehat{z})s_c(\widehat{z}) = \Lambda_1$). When $z > \widehat{z}$, the above first-order condition becomes

$$\zeta = zx(z)\widehat{\mathcal{P}}'(\lambda x(z)s_c(z)) = zx(z)\mathcal{P}'(\lambda x(z)s_c(z)) = zx(z)\cdot\lambda x(z)s_c(z)e^{-\lambda x(z)s_c(z)}. \tag{28}$$

Comparing the above equation with (7) implies that the derivative of rental price $r'(s) = \zeta$. The rental price in the coalition's solution is thus linear. The reason is that the marginal value of arc length should be equal to the gradient of location price (see equation (7)), which implies that the competitive price of locations must be linear in s. Therefore, an equivalent way of endogenizing the equilibrium location distribution is to allow sellers to buy and sell arc lengths at a unit price in a competitive market, so the total price is linear. The equilibrium unit price is such that the total demand of arc lengths equals the total supply.

We now analyze under what condition the planner's solution coincides with the coalition's solution. Since the planner's problem is strictly concave, its solution $s_p(z)$ is always continuous. As we showed above, the coalition's problem is not concave, so that its solution $s_c(z)$ can have a jump. To see this, recall that z_0 is the lowest seller type. If $z_0 = 0$ (or sufficiently small), then at the planner's solution there exists a threshold z' such that the optimal $s_p(z') = 0$ and $x(z)s_p(z)$ then increase strictly and continuously for $z > z'$. At the coalition's solution, there exists a threshold z'' such that $\lambda x(z'')s_c(z'') = \Lambda_1$ and $x(z)s_c(z)$ increases strictly and continuously for $z > z''$. In this case, the two solutions must differ. Therefore, a necessary condition for the planner's solution to coincide with the coalition's solution is that z_0 is sufficiently large so that $\lambda x(z_0)s_p(z_0) \geq \Lambda_1$. Conditional on this, the two solutions coincide if and only if the planner' solution also satisfies the coalition's first-order condition (28). Comparing (28) with the planner's first-order condition implies that the two solutions coincide if and only if $zx(z)$ is constant for all $z \geq z_0$. The following proposition summarizes the above result.

Proposition 3. *The planner's solution coincides with the coalition's solution (the decentralized equilibrium outcome) if and only if $zx(z) = z_0 x(z_0)$ for all $z \geq z_0$ and $\lambda x(z_0)z_0/\overline{z} \geq \Lambda_1$, where $\overline{z}$ is the mean of z ($\overline{z} = \int_{z_0}^{\infty} z dF(z)$). Furthermore, in this case $s(z) = z/\overline{z}$.*

Proof. See Appendix A.6. □

Suppose that the planner considers assigning additional Δs to a certain seller z. This increases surplus if and only if the seller meets no valuable buyers, in which case an additional buyer creates an expected surplus $zx(z)$. Since $zx(z)$ is constant, at the planner's optimum the probability that sellers meet no valuable buyers must be the same across different sellers. That is, the effective queue length $\lambda x(z)s_p(z)$ is constant across sellers. The coalition faces a different trade-off when considering to assign additional Δs to a certain seller z. It creates value for the seller if and only if the seller already meets with exactly one valuable buyer. In that case, an additional buyer creates an expected surplus $zx(z)$. In the special case that the effective queue length is constant across sellers, the probability that the sellers meet exactly one valuable buyer is also constant. Hence the marginal tradeoffs in terms of allocating arc length to sellers (intensive margin) for the planner and the coalition are exactly the same. However, the coalition would like to exclude some sellers from participation to take advantage of increasing returns to scale when the effective queue length is too small (we can think of determining who is active or not as adjustments along the extensive margin). If the measure of buyers is large enough, the coalition does not exclude sellers from the market. Then the two solutions coincide.

Next, we compare the planner's solution with the coalition's solution for the case where $zx(z)$ is strictly increasing.

Proposition 4. *Suppose that $zx(z)$ is strictly increasing. Then there exists some $z_1 \geq \widehat{z}$ such that $s_c(z) > s_p(z)$ for $z > z_1$, and $s_c(z) \leq s_p(z)$ for $z < z_1$ (the latter "$\leq$" holds as "$=$" only when $s_p(z) = 0$), where $\widehat{z}$ is the minimal type of active sellers in the coalition's solution.*

Proof. See Appendix A.7. □

When $zx(z)$ is strictly increasing, the planner will assign a longer effective queue length to sellers with a higher z such that the marginal contribution to surplus of arc lengths is the same across sellers. Consider two sellers a and b with $z_a < z_b$. At the planner's solution, the marginal value of arc-lengths must be the same across the two values: $z_a x(z_a) e^{-\widetilde{\lambda}_{p,a}} = z_b x(z_b) e^{-\widetilde{\lambda}_{p,b}}$, with effective queue lengths $\widetilde{\lambda}_{p,a} = \lambda x(z_a) s_p(z_a)$ and $\widetilde{\lambda}_{p,b} = \lambda x(z_b) s_p(z_b)$, which implies that $\widetilde{\lambda}_{p,a} < \widetilde{\lambda}_{p,b}$. From the coalition's point of view, the marginal value of arc length is higher at seller b in the planner's solution since $z_a x(z_a) \widetilde{\lambda}_a e^{-\widetilde{\lambda}_a} < z_b x(z_b) \widetilde{\lambda}_b e^{-\widetilde{\lambda}_b}$, since $\widetilde{\lambda}_b > \widetilde{\lambda}_a$. As a result, compared with the planner's solution, the coalition assigns a longer arc length to high-z sellers (more buyers has a relataively large effect on moving from one to two effective buyers) and a shorter arc length to low-z sellers, while the total arc length is fixed.

Comparison with Exogenous Location Distribution. In the last section where the location distribution was exogenous, we showed that PAM can fail in equilibrium if the products are sufficiently niche ($-1 \leq \varepsilon_x(z) < -1/2$), because sellers of high z do not find it worthwhile to pay for good locations if there are few buyers per seller. As we mentioned above, the difference between an exogenous and an endogenous distribution of locations is that in the latter case, the coalition (REIT) can not only choose the assignment between sellers and locations but also the distribution of locations. The latter case illustrates a new source of inefficiency. When the coalition has more choices, it will make sure that the number of buyers per seller is sufficiently large (potentially by excluding some sellers with low z from entry) so that the inefficiency from the case of exogenous distribution of locations does not arise and PAM always holds in equilibrium. To take advantage of niche products, the coalition will implement a distribution of locations that is too favorable for sellers of high z to ensure that the probability that sellers meet two or more effective buyers is sufficiently large. This channel is absent in the case where the location distribution is exogenous. We view the results from the two cases (exogenous versus endogenous distribution of locations) as complementary, since they illustrate different sources of inefficiencies.

4 Conclusion

In this paper, we developed a tractable search model that takes into account that some locations are better than others. We then use our framework to study what type of sellers benefit from good locations and how this translates into spatial sorting. Having heterogeneous regions leads to less trade but possibly to more quality-weighted trade. The resulting equilibrium can be inefficient because of a rent seeking externality. When there are few buyers per seller, sellers of general products benefit most from the good locations because when two or more buyers arrive, they receive the full surplus while for a social planner, one buyer who likes the product is enough and any additional buyer visit adds nothing to surplus. For sellers of niche products, the good spots, mainly increase the probability of one buyer who likes their product but the probability of two effective buyer arrivals remains close to zero (in this case with few buyers per seller).

In our model, we defined x to be the probability that a buyer likes the product but alternatively, we can think of x as the probability that a credit-constrained buyer can afford the product. Within cities, sellers of expensive high-quality products would still have incentives to move to good locations. At those good locations, expensive niche products will be offered that the poor cannot afford. In the other areas, more standard goods will be offered. The buyers and sellers in other areas (with fewer buyers per seller) will trade more standard products (H&M, McDonalds) but they can also benefit because those areas will be cheaper. The welfare effects also depend on who receives the rents from the good locations, but we leave a more detailed analysis of that for future research.

Another novel implication of our model is that both the aggregate meeting and matching function, which maps the measures of buyers and sellers into respectively meetings and trades, is ultimately driven by the distribution of product quality. Finally, we showed that when the location distribution is endogeneous, the market outcome need not be efficient because sometimes sellers invest too much in good locations in order to increase the likelihood of multiple buyers which increases the price.

A natural extension of our framework would be to allow for other pricing mechanisms. We have focused on sellers posting auctions and showed that similar results are obtained with price posting with heterogeneous buyers. Another way to model price posting is to allow some buyers to contact multiple sellers as in Burdett and Judd (1983) or to account for directed search as in Burdett et al. (2001). Both alternatives make the analysis considerably more complicated and are left for future research.

Appendix A Additional Results and Omitted Proofs

A.1 Proof of Lemma 1

Since $S(s, z)$ is given by equation (2), we have

$$\frac{\partial S(s,z)}{\partial s} = zx(z)\lambda e^{-\lambda s x(z)} > 0$$

Similarly,

$$\begin{aligned}\frac{\partial S(s,z)}{\partial z} &= e^{-\lambda s x(z)}\left(\lambda s z x'(z) - 1 + e^{\lambda s x(z)}\right) > e^{-\lambda s x(z)}\left(\lambda s z x'(z) - 1 + 1 + \lambda s x(z)\right) \\ &= e^{-\lambda s x(z)}\lambda s x(z)\left(\frac{zx'(z)}{x(z)} + 1\right) \geq 0\end{aligned}$$

where for the last inequality we used the fact that $zx(z)$ is weakly increasing. Furthermore,

$$\frac{\partial^2 S(s,z)}{\partial s \partial z} = \lambda e^{\lambda(-s)x(z)}\left(1 + \frac{zx'(z)}{x(z)} - \frac{zx'(z)}{x(z)}\lambda s x(z)\right) \geq \lambda e^{\lambda(-s)x(z)}\left(-\frac{zx'(z)}{x(z)}\lambda s x(z)\right) \geq 0$$

where for the first inequality we used the fact that $zx(z)$ is weakly increasing and for the second inequality that $x(z)$ is weakly decreasing. Apparently, the two weak inequalities can not hold with equality at the same time, which implies that $S(s, z)$ is *strictly* supermodular. □

A.2 Proof of Lemma 2

Since $\pi(s, z)$ is given by equation (3) and equals $z\mathcal{P}(\lambda s x(z))$, we have

$$\frac{\partial \pi(s,z)}{\partial s} = \lambda z x(z)\mathcal{P}'(\lambda s x(z)) = \lambda^2 s z x(z)^2 e^{-\lambda s x(z)} > 0$$

Similarly,

$$\frac{\partial \pi(s,z)}{\partial z} = \mathcal{P}(\lambda s x(z))\left(1 + \varepsilon_x(z)\varepsilon_{\mathcal{P}}(\lambda s x(z))\right)$$

where $\varepsilon_x(z) = zx'(z)/x(z) \in [-1, 0]$ and $\varepsilon_{\mathcal{P}}(\lambda)$ is given by

$$\varepsilon_{\mathcal{P}}(\lambda) = \frac{\lambda \mathcal{P}'(\lambda)}{\mathcal{P}(\lambda)} = \frac{\lambda^2}{e^\lambda - \lambda - 1} > 0.$$

Note that $\lim_{\lambda\to 0}\varepsilon_{\mathcal{P}}(\lambda) = 2$ and $\lim_{\lambda\to\infty}\varepsilon_{\mathcal{P}}(\lambda) = 0$. Furthermore,

$$\varepsilon'_{\mathcal{P}}(\lambda) = -\frac{\lambda\left(e^{\lambda}(\lambda-2)+\lambda+2\right)}{\left(e^{\lambda}-\lambda-1\right)^2} < 0,$$

where the inequality follows from the fact that $e^{\lambda}(\lambda-2)+\lambda+2$ is a convex function and its derivative at $\lambda = 0$ equals 0. Hence, $\varepsilon_{\mathcal{P}}(\lambda)$ is strictly decreasing. Let $\Lambda(\cdot)$ be the inverse function of $\varepsilon_{\mathcal{P}}(\lambda)$. If $\varepsilon_x(z) \geq -1/2$, then $\pi_z(s,z) > 0$ since $1+\varepsilon_x(z)\varepsilon_{\mathcal{P}}(\lambda s x(z))$ is always strictly positive; if $\varepsilon_x(z) \in [-1,-1/2)$, then $\pi_z(s,z) > 0$ if and only if $\lambda s x(z) > \Lambda(-1/\varepsilon_x(z))$.

Next, we have

$$\frac{\partial^2 \pi(s,z)}{\partial s \partial z} = \lambda^2 s x(z)^2 e^{-\lambda s x(z)}\left(1+(2-\lambda x(z)s)\,\varepsilon_x(z)\right) \tag{29}$$

Since $\varepsilon_x(z) \leq 0$, the above is strictly positive if $\varepsilon_x(z) \geq -1/2$. If $\varepsilon_x(z) \in [-1,-1/2)$, then the above is strictly positive if and only if $\lambda x(z)s > 2+1/\varepsilon_x(z)$.

Finally, suppose that $\varepsilon_x(z) \in [-1,-1/2)$. Note that $\Lambda(-1/\varepsilon_x(z)) > 2+1/\varepsilon_x(z)$ if and only if $\varepsilon_{\mathcal{P}}(\lambda) > 2-\lambda$, where $\lambda = 2+1/\varepsilon_x(z) \in [1,2)$. Note that $\varepsilon_{\mathcal{P}}(0) = 2$. We now prove that $\varepsilon_{\mathcal{P}}(\lambda)+\lambda$ is strictly increasing. To see this, note that $\varepsilon'_{\mathcal{P}}(\lambda)+1 > 0$ if and only if $0 < e^{\lambda}-1-\lambda e^{\lambda/2}$, which holds because the right-hand side is strictly increasing: its derivative is given by $e^{\lambda/2}(e^{\lambda/2}-1-\lambda/2)$, which is greater than zero. □

A.3 Proof of Lemma 3

Since $Y(\lambda,\alpha,s_0)$ is given by equation (12),

$$\frac{\partial Y(\lambda,\alpha,s_0)}{\partial \alpha} = \overline{z}\frac{-(1-s_0)\lambda e^{-\lambda s_0}}{\left((\alpha-1)+\alpha\lambda(1-s_0)\right)^2} < 0,$$

where $s_0 < 1$. Note that in the above analysis, we change the quality distribution while holding fixed the mean quality $\overline{z}$ and the measure of buyers.

Similarly, we have

$$\frac{\partial Y(\lambda,\alpha,s_0)}{\partial s_0} = \overline{z}\frac{(\alpha-1)\lambda e^{-\lambda s_0}}{\left((\alpha-1)+\alpha\lambda(1-s_0)\right)^2}\left((1-s_0)\alpha\lambda-1\right).$$

The sign of $\partial Y(\lambda,\alpha,s_0)/\partial s_0$ is thus completely determined by the term $(1-s_0)\alpha\lambda-1$. When $\alpha\lambda \leq 1$, total surplus is maximal at $s_0 = 0$. When $\alpha\lambda > 1$, then total output is maximal at $s_0 = 1-1/(\alpha\lambda)$. □

A.4 Proof of Proposition 2

Consider the case where $s_0 = 0$. Normalize $z_0 = 1$. First, we calculate the price of locations explicitly.[19] Plugging $z^*(s)$ from equation (11) into equation (7) yields,

$$r_0'(s) = \lambda^2 s e^{-s(\lambda - 1/\alpha)}.$$

where the subscript 0 denotes the case $s_0 = 0$. Solving the above differential equation yields:

$$r_0(s) = \left(1 + \frac{1}{\alpha(\lambda - \frac{1}{\alpha})}\right)^2 \left(1 - e^{-s(\lambda - \frac{1}{\alpha})} - s(\lambda - \frac{1}{\alpha})e^{-s(\lambda - \frac{1}{\alpha})}\right).$$

The average location price is given by

$$R_0 = \int_0^\infty r_0(s)\, d(1 - e^{-s}) = \left(\frac{\alpha\lambda}{\alpha\lambda + \alpha - 1}\right)^2.$$

Since we assume that $\lambda\alpha = 1$, the above two equations imply $r(s) = s^2/(2\alpha^2)$ and $R_0 = 1/\alpha^2$. By equation (11), we have $s^*(z) = \alpha \log(z)$. Plugging $s^*(z)$, $r_0(s^*(z))$ and R_0 into equation (13) then yields

$$\Pi_0(z) = z - \frac{1}{2}\log(z)(\log(z) + 2) - 1 + \frac{1}{\alpha^2}$$

which implies that $\lim_{z \to 0} \Pi_0'(z) = 1$ and $\Pi_0''(z) = \log(z)/z^2 > 0$, since $z > z_0 = 1$.

Define $\Delta\Pi(z) = \Pi_0(z) - \Pi_1(z)$. Since $\Pi_0(z)$ is strictly convex and $\Pi_1(z)$ is linear, $\Delta\Pi(z)$ is strictly convex. Next, consider the derivative of $\Delta\Pi(z)$:

$$\Delta\Pi'(z) = \frac{e^{-1/\alpha}(\alpha + 1)}{\alpha} - \frac{\log(z) + 1}{z}, \tag{30}$$

where we used the assumption $\lambda = 1/\alpha$. The second term on the right-hand side is strictly decreasing in z, since its derivative equals $-\log(z)/z^2 < 0$. Furthermore, the first term on the right-hand side is strictly smaller than 1 since $e^{1/\alpha} > 1 + 1/\alpha$. Thus $\Delta\Pi(z)$ reaches its minimum at z_m where $\Delta\Pi'(z_m) = 0$. Note that

$$\Delta\Pi(z_m) = \frac{1}{\alpha^2} - \frac{(\log z_m)^2}{2}$$

[19] When $s_0 \in (0, 1)$, $r(s)$ can also be calculated explicitly with a more complicated expression.

which implies that $\Delta\Pi(z_m) > 0$ if and only if $z_m < e^{\sqrt{2}/\alpha}$, which, by (30), holds if and only if

$$\frac{e^{-1/\alpha}(\alpha+1)}{\alpha} > \frac{\log(e^{\sqrt{2}/\alpha})+1}{e^{\sqrt{2}/\alpha}}$$

which is equivalent to

$$e^{\frac{\sqrt{2}-1}{\alpha}} - \left(1 + \frac{\sqrt{2}-1}{1+\alpha}\right) > 0$$

The left-hand side above is decreasing in α, since its derivative equals $\left(\sqrt{2}-1\right)\alpha^2 \left(\frac{\alpha^2}{(\alpha+1)^2} - e^{\frac{\sqrt{2}-1}{\alpha}}\right) < 0$. Furthermore, when $\alpha \to \infty$, it approaches to zero. Therefore, the above inequality holds, and we have proved our claim. □

A.5 Proof of Equation (20)

Since $P_n(\lambda)$ is given by equation (16), we have

$$\begin{aligned} 1 - \sum_{n=0}^{\infty} P_n(\lambda)(1-x)^n &= 1 - \sum_{n=0}^{\infty} \left(\int_s e^{-\lambda s} \frac{(\lambda s)^n}{n!} \, dL(s) \right) (1-x)^n \\ &= 1 - \int_s \left(\sum_{n=0}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n (1-x)^n}{n!} \right) dL(s) = 1 - \int_s e^{-\lambda x s} \, dL(s) = \int_s 1 - e^{-\lambda x s} \, dL(s) \end{aligned}$$

where for the first equality in the second row, we interchanged summation and integration, and the second equality follows from the fact that the Poisson probabilities sum up to 1. The last term on the right-hand side is exactly $m(\lambda x)$ by definition. □

A.6 Proof of Proposition 3

By equation (23), when $s(z) = 0$, the marginal benefit of increasing $s(z)$ is $zx(z)$. Therefore, all sellers should be active ($s(z) > 0$). Furthermore, the first-order condition (23) implies that $x(z)s_p(z)$ should be constant across different seller types, which implies that $s_p(z)$ must be proportional to z. Since the mean of s is 1, $s_p(z)$ is simply z divided by the mean of z: $s_p(z) = z/\overline{z}$, where $\overline{z} = \int_{z_0}^{\infty} z dF(z)$.

As we argued before Proposition 3, a necessary condition for the planner's solution to coincide with the coalition's solution is that $\lambda x(z_0)s_p(z_0) > \Lambda_1$. Furthermore, since $s_p(z)$ needs to also satisfy equation (28), comparing (28) with the planner's FOC (23) shows that $zx(z)$ needs to be constant for $z \geq z_0$. If the above two necessary conditions are satisfied, they are also sufficient since the first-order conditions are not only necessary but also sufficient.□

A.7 Proof of Proposition 4

The planner's solution $s_p(z)$ is a continuously increasing function which is determined by the first-order condition (23): $\mathcal{F}^p(s,z) = \xi$, where $\mathcal{F}^p(s,z) = zx(z)e^{-\lambda x(z)s}$. The coalition's solution $s_c(z)$ has one possible jump point $\widehat{z}$ below which $s_c(z)$ is zero and above which $s_c(z)$ is continuously increasing with $\lambda x(z)s_c(z) \geq \Lambda_1$ and it is given by the coalition's first-order condition (28): $\mathcal{F}^c(s,z) = \zeta$, where $\mathcal{F}^c(s,z) = zx(z)\lambda x(z)se^{-\lambda x(z)s}$. Of course, it may be the case that $\widehat{z}$ does not exist in which case we set $\widehat{z}$ to be z_0.

Next, we show that the level curves of $\mathcal{F}^p(s,z)$ and $\mathcal{F}^c(s,z)$ satisfy the single-crossing property.

$$-\frac{\partial\mathcal{F}^c(s,z)/\partial z}{\partial\mathcal{F}^c(s,z)/\partial s} - \left(-\frac{\partial\mathcal{F}^p(s,z)/\partial z}{\partial\mathcal{F}^p(s,z)/\partial s}\right) = \frac{1+\varepsilon_x(z)}{\lambda zx(z)\left(\lambda x(z)s-1\right)}$$

where $\varepsilon_x(z) = zx'(z)/x(z) > -1$ (we assume that $zx(z)$ is strictly increasing). Since at the coalition's interior solution, $\lambda x(z)s \geq \Lambda_1 > 1$, the above equation is always strictly positive.

If $\widehat{z} = z_0$, then $s_c(z)$ is continuous for $z \geq z_0$. Since $1 = \int_z s_p(z)dF(z) = \int_z s_c(z)dF(z)$, by continuity there exists some $z_1 > z_0$ such that $s_p(z_1) = s_c(z_1)$, denote which by s_1. As we showed above, the level curve $\mathcal{F}^p(s,z) = \mathcal{F}^p(s_1,z_1)$ crosses the level curve $\mathcal{F}^c(s,z) = \mathcal{F}^c(s_1,z_1)$ once and from below. Thus $s_c(z) > s_p(z)$ for $z > z_1$ and $s_c(z) < s_p(z)$ for $z < z_1$.

Next, suppose that $\widehat{z} > z_0$. If $s_c(\widehat{z}) \geq s_p(\widehat{z})$, then $s_c(z) > s_p(z)$ for all $z > \widehat{z}$, otherwise the level curve $\mathcal{F}^c(s,z) = \zeta$ must cross the level curve $\mathcal{F}^p(s,z) = \xi$ from above at some point. If $s_c(\widehat{z}) < s_p(\widehat{z})$, by continuity there exists some $z_1 > \widehat{z}$ such that $s_p(z_1) = s_c(z_1)$ since $1 = \int_{z\geq\widehat{z}} s_c(z)dF(z) \geq \int_{z\geq\widehat{z}} s_p(z)dF(z)$. Again by the single-crossing property, $s_c(z) > s_p(z)$ for $z > z_1$ and $s_c(z) < s_p(z)$ for $\widehat{z} \leq z < z_1$. When $z < \widehat{z}$, $s_c(z) = 0$ so that $s_c(z) \leq s_p(z)$ continuous to hold, and in this case $s_c(z) = s_p(z)$ if and only if $s_p(z) = 0$. □

References

Albrecht, J., Menzio, G., and Vroman, S. (2023). Vertical differentiation in frictional product markets. *Journal of Political Economy Macroeconomics*, forthcoming.

Becker, G. S. (1973). A theory of marriage: Part I. *Journal of Political Economy*, 81(4):813–846.

Becker, L. and Mangin, S. (2023). Extreme value theory with heterogeneous agents. mimeo.

Burdett, K. and Judd, K. L. (1983). Equilibrium price dispersion. *Econometrica*, 51:955–969.

Burdett, K., Shi, S., and Wright, R. (2001). Pricing and matching with frictions. *Journal of Political Economy*, 109:1060–1085.

Cai, X., Gautier, P. A., and Wolthoff, R. P. (2017). Search frictions, competing mechanisms and optimal market segmentation. *Journal of Economic Theory*, 169:453 – 473.

Combes, P.-P., Duranton, G., and Gobillon, L. (2008). Spatial wage disparities: Sorting matters! *Journal of Urban Economics*, 63(2):723–742.

Dauth, W., Findeisen, S., Moretti, E., and Suedekum, J. (2022). Matching in Cities. *Journal of the European Economic Association*, 20(4):1478–1521.

Davis, S. J. and de la Parra, B. S. (2017). Application flows. mimeo.

Ellison, G. and Glaeser, E. L. (1997). Geographic concentration in u.s. manufacturing industries: A dartboard approach. *Journal of Political Economy*, 105(5):889–927.

Ellison, G., Glaeser, E. L., and Kerr, W. R. (2010). What causes industry agglomeration? evidence from coagglomeration patterns. *American Economic Review*, 100(3):1195–1213.

Fujita, M. and Thisse, J.-F. (2002). *Economics of Agglomeration: Cities, Industrial Location, and Regional Growth*. Cambridge University Press.

Gautier, P. A., Svarer, M., and Teulings, C. N. (2010). Marriage and the city: Search frictions and sorting of singles. *Journal of Urban Economics*, 67(2):206–218.

Gautier, P. A. and Teulings, C. N. (2003). An Empirical Index for Labor Market Density. *The Review of Economics and Statistics*, 85(4):901–908.

Gautier, P. A. and Teulings, C. N. (2009). Search and the city. *Regional Science and Urban Economics*, 39(3):251–265.

Helsley, R. W. and Strange, W. C. (1990). Matching and agglomeration economies in a system of cities. *Regional Science and Urban Economics*, 20(2):189–212.

Kim, S. (1989). Labor specialization and the extent of the market. *Journal of Political Economy*, 97(3):692–705.

Kiyotaki, N. and Wright, R. (1993). A search-theoretic approach to monetary economics. *American Economic Review*, 83(1):63–77.

Lester, B., Visschers, L., and Wolthoff, R. (2015). Meeting technologies and optimal trading mechanisms in competitive search markets. *Journal of Economic Theory*, 155:1–15.

Menzio, G. (2023). Optimal product design: Implications for competition and growth under declining search frictions. *Econometrica*, 91(2):605–639.

Moretti, E. (2012). *The new geography of jobs*. Houghton Mifflin Harcourt.

Neiman, B. and Vavra, J. (2023). The rise of niche consumption. *American Economic Journal: Macroeconomics*, forthcoming.

Pissarides, C. A. (2000). *Equilibrium Unemployment Theory*. Oxford University Press.

Rosenthal, S. S. and Strange, W. C. (2004). Chapter 49 - evidence on the nature and sources of agglomeration economies. In Henderson, J. V. and Thisse, J.-F., editors, *Cities and Geography*, volume 4 of *Handbook of Regional and Urban Economics*, pages 2119–2171. Elsevier.